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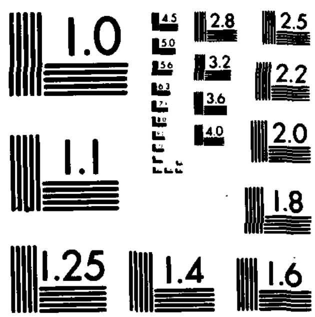

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Two Exponential Approximation Methods

Roy L. Streit
Information Services Department

28 October 1980

NUSC
Naval Underwater Systems Center
Newport, Rhode Island • New London, Connecticut

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Preface

The research reported herein was conducted under NUSC Project No. A70210, *Optimization of Mutually Coupled Arrays*, Principal Investigatpr. R. L. Streit (Code 7122).

The Technical Reviewer for this report was D. H. Wood (Code 3342).

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Paley-Wiener Theorem
Bernstein's Theorem
Completely Monotonic Functions

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Two different constructive techniques for approximating positive definite functions by means of finite exponential sums are explored. One technique constructs the coefficients and the exponents. The other technique constructs the exponents when the coefficients are all required to be equal. Both approximation techniques appear to be suitable for numerical computation. The techniques extend to completely monotonic functions as well. Error bounds are proved using elementary methods.

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In an application, these error bounds can be used to eliminate some of the effort and guesswork previously necessary in two procedures for the design and synthesis of sparse broadband linear arrays.

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Two Exponential Approximation Methods

I. Introduction

Two design procedures for aperiodic, or space tapered, linear arrays are investigated in this report in a setting much more general than the usual setting. One procedure, due to Bruce and Unz [1], gives both element excitations ("shadings") and positions. The other procedure, due to Maffett [2], gives element positions under the condition that all excitations are unity. Both seek desirable radiation patterns minimizing grating lobes. These methods synthesize sparse broadband arrays that are less sensitive to frequency changes than periodic (equispaced) arrays.

Using either of these procedures, the designer must guess the number of elements required, perform the appropriate numerical computations, examine the resulting radiation pattern, and then decide if more elements are required or if fewer elements will suffice. In this report, error bounds are derived that provide estimates on the number of elements necessary for a given degree of approximation of the desired radiation pattern. Thus, some of the effort and guesswork inherent in these procedures can be eliminated.

Neither of these two methods is intrinsically limited to aperiodic array design. Generalizations turn out to be worthwhile and of independent interest. Therefore, this report addresses only the general setting from this point on.

A complex valued function f of a real variable is defined to be positive definite if and only if, for each integer $n \geq 1$, the inequality

$$\sum_{i,j=1}^{n} f(x_i - x_j)\, a_i\, \bar{a}_j \geq 0 \qquad (1.1)$$

holds for all $x_1, \ldots, x_n \, \varepsilon \mathbf{R}$ (the real numbers) and $a_1, \ldots, a_n \, \varepsilon \mathbf{C}$ (the complex numbers). Bochner's Theorem states the following: If f is a continuous function on **R**, then f is positive definite if, and only if, there exists a bounded non-decreasing function V on **R** such that f is the Fourier-Stieltjes transform of V; that is,

$$f(x) = \int_{-\infty}^{\infty} e^{i\alpha x}\, dV(\alpha), \qquad x \varepsilon \mathbf{R} \, . \qquad (1.2)$$

The recent paper of Stewart [3] gives references to various proofs of Bochner's Theorem and its generalizations. We point out, for future use, that (1.2) immediately implies that the total variation of V equals f(0), and for all real x, f(−x) equals the complex conjugate of f(x). Goldberg [4] proves that any positive definite function f is such that $|f(x)| \leq f(0)$ for all real x.

In this report, we restrict our attention, for the most part, to continuous positive definite functions f on **R** that can be written

$$f(x) = \int_{-\lambda}^{\lambda} e^{i\alpha x}\, dV(\alpha), \qquad x \varepsilon \mathbf{R}, \qquad (1.3)$$

for some real number λ such that $0 < \lambda < \infty$. In other words, we have assumed that $V(\alpha)$ is constant for $|\alpha| > \lambda$. For functions satisfying (1.3), we develop in an elementary manner an approximation to $f(x)$ of the form

$$S_n(x) = f(0) \sum_{k=1}^{n} a_k e^{i\alpha_k x}, \tag{1.4}$$

where $|\alpha_k| < \lambda$, $k = 1, \ldots, n$. We give an error bound for this approximation in Theorem 1. This approximation always gives positive coefficients and exponents $\alpha_k \epsilon \mathbf{R}$ that are located at the roots of an appropriate orthogonal polynomial. We suspect that these approximations are near-optimal in some well-defined sense. (See Schabach [5, p. 1018] for a relevant conjecture about a particular function f.)

Under various additional assumptions concerning V, we develop an approximation to $f(x)$ of the form

$$Q_n(x) = \frac{f(0)}{n} \sum_{k=1}^{n} e^{i\alpha_k x}, \tag{1.5}$$

where $|\alpha_k| < \lambda$, $k = 1, \ldots n$, and we give an error bound in Theorem 2. The approximation $Q_n(x)$ cannot be as efficient in general as the approximation $S_n(x)$; however, $Q_n(x)$ has the advantage of being much more easily constructed in practice for almost any reasonable n (say, $n < 10^6$).

Note that both the approximations $S_n(x)$ and $Q_n(x)$ are readily written in the form (1.3) and, therefore, are positive definite. Hence, we must have

$$|f(x)-S_n(x)| \leqslant |f(x)| + |S_n(x)| \leqslant 2f(0), \qquad x\epsilon\mathbf{R}, \tag{1.6}$$

since $S_n(0) = f(0)$. Similarly, it is always the case that $|f(x) - Q_n(x)| \leqslant 2\, f(0)$.

It will be shown that Prony's method can be used to compute $S_n(x)$. Although Prony's method in this problem must become numerically ill-conditioned for n sufficiently large, it may nonetheless be useful for small n (say, $n \leqslant 10$). Numerically stable methods for computing $S_n(x)$ suitable for all n would require an algorithm other than Prony's method. This is discussed at the end of Section II.

The computation of approximations $Q_n(x)$ is shown to depend upon the ability to compute the numerical value of the inverse function of V (guaranteed to exist by additional assumptions) at specific points. The level of difficulty involved depends on V, of course, but the interval is finite, so the problem seems to encounter no inherent numerical difficulties.

An excellent bibliography of references to the literature on exponential approximation is contained in [6].

Note that if V in (1.3) is continuously differentiable on the interval $(-\lambda, \lambda)$, then $V'(\alpha) \geqslant 0$, and

$$f(x) = \int_{-\lambda}^{\lambda} e^{i\alpha x} V'(\alpha)\, d\alpha . \tag{1.7}$$

From the Paley-Wiener Theorem (see, e.g., [7, p. 134]), this equation uniquely extends the domain of f to all $\mathbf{C}$ and that this extension of f is an entire function of exponential type at most λ.

We close this section with a small collection of positive definite functions. According to [3], Schoenberg proved that $f_r(x) = \exp[-|x|^r]$ is positive definite if and only if $0 \leq r \leq 2$, and Polya proved that any real, even, continuous function f that is convex on the interval $(0, \infty)$, that is, $f((x+y)/2) \leq (f(x)+f(y))/2$, and satisfies $\lim_{x \to \infty} f(x) = 0$, is positive definite. Goldberg [4, p. 61] proves that if f(x) is positive definite and $a > 0$, then the function $h(x) = f(x) \exp(-ax^2)$ is also positive definite. Finally, if the function f has a Fourier transform that is nonnegative and integrable, then the function f is positive definite. Specific examples of functions satisfying this latter property are

$$\frac{\sin \lambda x}{x} = \frac{1}{2}\int_{-\lambda}^{\lambda} e^{i\alpha x}\, d\alpha \,, \tag{1.8}$$

$$\frac{2(1-\cos \lambda x)}{\lambda x^2} = \int_{-\lambda}^{\lambda} e^{i\alpha x}\left(1 - \frac{|\alpha|}{\lambda}\right) d\alpha \tag{1.9}$$

$$e^{-|x|} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{1+\alpha^2}\, d\alpha \tag{1.10}$$

$$e^{-ax^2} = \frac{1}{\sqrt{4a\pi}}\int_{-\infty}^{\infty} e^{i\alpha x}\, e^{-\alpha^2/4a}\, d\alpha \qquad (a > 0) \tag{1.11}$$

$$\frac{J_\nu(\lambda x)}{x^\nu} = \frac{(2\lambda)^{-\nu}}{\pi^{1/2}\Gamma(\nu+\frac{1}{2})}\int_{-\lambda}^{\lambda} (\lambda^2 - \alpha^2)^{\nu-1/2}\, e^{i\alpha x}\, d\alpha \qquad (\nu > -\tfrac{1}{2}) \tag{1.12}$$

where $J_\nu(x)$ is the usual Bessel function of order ν. A final example, one that finds application in antenna design ([8], [9], and [10]), is

$$\cos\left(\sqrt{(\lambda x)^2 - a^2}\,\right) = \int_{-\lambda}^{\lambda} e^{i\alpha x}\, dV(\alpha) \,, \qquad a \geq 0, \tag{1.13}$$

where $V(-\lambda) = -1/2\lambda$,

$$V(\alpha) = \frac{a}{2}\int_{-\lambda}^{\alpha} \frac{I_1\left(\frac{a}{\lambda}\sqrt{\lambda^2 - t^2}\,\right)}{(\lambda^2 - t^2)^{1/2}}\, dt, \qquad -\lambda < \alpha < \lambda \,,$$

$V(+\lambda) = 1/2\lambda + \lim_{\alpha \to \lambda^-} V(\alpha)$, and where $I_1(x)$ is the modified Bessel function of order one. This function is interesting because, for $|x| \geq a/\lambda$, it has magnitude not exceeding 1, while for $|x| < a/\lambda$, it exhibits very rapid growth achieving a maximum magnitude of cosh(a) at $x = 0$. Other examples can be discerned in various tables of integral transforms, such as [11].

II. Exponential Approximation With Arbitrary Coefficients

The idea developed in this section for constructing approximations of the form $S_n(x)$ is simply Gaussian quadrature. A glance at equation (1.3) reveals that we are particularly interested in Gaussian quadrature with respect to the measure $dV(\alpha)$. From Szegö [12, p. 25], a system of orthogonal polynomials exists for the measure $dV(\alpha)$ if $V(\alpha)$ has infinitely many points of increase in the interval $[-\lambda, -\lambda]$ and if the moments

$$c_m = \int_{-\lambda}^{+\lambda} \alpha^m \, dV(\alpha) \, , \; m = 0,1,2, \ldots \tag{2.1}$$

exist. Since V is bounded above, the moments c_m certainly exist. If V has finitely many points of increase, then f can be written explicitly as a finite sum of exponentials. Although this special case is not uninteresting (in the context of economizing large finite exponential sums), we will avoid it by assuming that V has infinitely many points of increase.

Let $\alpha_1, \ldots, \alpha_n$ be the abscissas and let $b_1, \ldots, b_n$ be the corresponding Cotes numbers of the n-th order Gaussian quadrature formula with respect to the measure $dV(\alpha)$. Since

$$\sum_{k=1}^{n} b_k = \int_{-\lambda}^{\lambda} 1 \, dV(\alpha) = f(0) \, ,$$

we rewrite the Cotes numbers in the form $b_k = a_k f(0)$, $k = 1, \ldots, n$. Using this notation, and applying the quadrature formula blindly to (1.3) gives the approximation

$$S_n(x) = f(0) \sum_{k=1}^{n} a_k e^{i\alpha_k x} \, , \tag{2.2}$$

where

$$0 < a_k < 1, \; k = 1, \ldots, n \tag{2.3}$$

$$a_1 + \ldots + a_n = 1 \tag{2.4}$$

$$|\alpha_k| < \lambda, \; k = 1, \ldots, n \, . \tag{2.5}$$

These three properties are immediate consequences of well known results on Gaussian quadrature. (See Szegö [12, pp. 47-49].) In addition, these properties imply that $S_n(x)$ possesses good numerical round-off error behavior when the sum is evaluated numerically.

We seek an error bound such that

$$|f(x) - S_n(x)| \leq R_n(x) \, , \qquad x \epsilon \mathbf{R} \, . \tag{2.6}$$

It is clear from (1.3) and the Riemann-Lebesgue Lemma that $f(x) \to 0$ as $x \to \infty$. On the other hand, it is not hard to see from (2.2) that $S_n(x)$ cannot tend to zero as $x \to \infty$. The most that can be expected is that $R_n(x)$ becomes "small" for any fixed x. We will show that as $n \to \infty$, S_n converges to f uniformly on any finite real interval.

Let $n \geq 1$. For each $x \varepsilon \mathbf{R}$, let

$$\varepsilon_n(x) = \min \max_{-\lambda \leq \alpha \leq \lambda} | e^{i\alpha x} - \pi_{2n-1}(\alpha)| , \tag{2.7}$$

where the minimum is taken over all polynomials $\pi_{2n-1}(\alpha)$ of degree at most 2n-1 with complex coefficients. We always have $\varepsilon_n(x) \leq 1$ for all x, as can be seen by considering the case $\pi_{2n-1}(\alpha) \equiv 0$ in (2.7).

Lemma 1. For $n \geq 1$,

$$\varepsilon_n(x) \leq \sqrt{2} \, \frac{(\lambda x)^{2n}}{2^{2n-1}(2n)!} , \quad x \varepsilon \mathbf{R} . \tag{2.8}$$

Proof. From a theorem given in [13, p.78], for any real valued function $p(\alpha)$ defined on the interval $[-1, +1]$ and possessing $n+1$ continuous derivatives on $(-1, +1)$, we have

$$E_n(p) \equiv \min \max_{-1 \leq \alpha \leq 1} | p(\alpha) - \pi_n(\alpha) | = \frac{|p^{(n+1)}(\zeta)|}{2^n(n+1)!}$$

for some ζ, $-1 < \zeta < +1$, where the minimum is taken over all real polynomials π_n of degree at most n. For $p(\alpha) = \cos \alpha\lambda x$ defined for α in the interval $[-1, 1]$ and for fixed real numbers λ and x, we have

$$\begin{aligned} E_{2n-1}(p) &= \frac{(\lambda x)^{2n}}{2^{2n-1}(2n)!} |\cos \zeta\lambda x| \\ &\leq \frac{(\lambda x)^{2n}}{2^{2n-1}(2n)!} . \end{aligned}$$

For $q(\alpha) = \sin \alpha\lambda x$ on $[-1, 1]$, we have similarly

$$E_{2n-1}(q) \leq \frac{(\lambda x)^{2n}}{2^{2n-1}(2n)!} .$$

From the definition of $\varepsilon_n(x)$, we have

$$\begin{aligned} \varepsilon_n(x) &= \min \max_{-1 \leq \alpha \leq 1} |e^{i\alpha\lambda x} - \pi_{2n-1}(\lambda\alpha)| \\ &\leq \{E_{2n-1}^2(p) + E_{2n-1}^2(q)\}^{1/2} . \end{aligned}$$

Substituting the estimates for $E_{2n-1}(p)$ and $E_{2n-1}(q)$ completes the proof.

We remark, but do not prove, that an example in Meinardus [13, p. 96] can be extended and used to show that for fixed x,

$$\varepsilon_n(x) \leq \frac{(\lambda x)^{2n}}{2^{2n-1}(2n)!} (1 + o(1)) , n \to \infty .$$

It seems reasonable to conjecture that this asymptotic inequality is actually an asymptotic equality. In any event, we use only (2.8) in this report.

Theorem 1. Let f(x) be a continuous complex valued positive definite function of a real variable such that

$$f(x) = \int_{-\lambda}^{\lambda} e^{i\alpha x}\, dV(\alpha)\ , \qquad x\epsilon\mathbf{R}, \qquad (2.9)$$

where V is a bounded non-decreasing function having infinitely many points of increase in the finite closed interval $[-\lambda, \lambda]$. Then, for each integer $n \geq 1$, there exists distinct real numbers $\alpha_1, \ldots, \alpha_n$ and real numbers $a_1, \ldots, a_n$ satisfying (2.3), (2.4), and (2.5) and the additional condition

$$|f(x) - S_n(x)| \leq \sqrt{2}\ f(0)\ \frac{(\lambda x)^{2n}}{2^{2n-2}(2n)!}\ , \ x\epsilon\mathbf{R}\ , \qquad (2.10)$$

where $S_n(x)$ is given by (2.2). Furthermore, the left-hand side of (2.10) is never larger than 2f(0) for all $x\epsilon\mathbf{R}$ and every integer $n \geq 1$.

Proof. Let $\alpha_1, \ldots, \alpha_n$ be the distinct abscissas of an n point Gaussian quadrature formula with respect to the measure $dV(\alpha)$, and let $b_1, \ldots, b_n$ be the corresponding Cotes numbers. Let the numbers $a_1, \ldots, a_n$ be defined by the relationship $b_k = f(0)\, a_k$, $k = 1, \ldots, n$. Equations (2.3), (2.4), and (2.5) are then satisfied. Fix $x\epsilon\mathbf{R}$. Let $p^*(\alpha)$ be any polynomial of degree at most 2n–1 such that

$$\max_{-\lambda \leq \alpha \leq \lambda} |e^{i\alpha x} - p^*(\alpha)| = \varepsilon_n(x)\ ,$$

where $\varepsilon_n(x)$ is defined by (2.7). Then, defining $S_n(x)$ as in (2.2), we have

$$\begin{aligned}
|f(x) - S_n(x)| &\leq |f(x) - f(0) \sum_{k=1}^{n} a_k\, p^*(\alpha_k)| \\
&\quad + |f(0) \sum_{k=1}^{n} a_k\, p^*(\alpha_k) - S_n(x)| \\
&\leq \int_{-\lambda}^{\lambda} |e^{i\alpha x} - p^*(\alpha)|\, dV(\alpha) \\
&\quad + f(0) \sum_{k=1}^{n} a_k\, |p^*(\alpha_k) - e^{i\alpha_k x}| \\
&\leq \varepsilon_n(x) \int_{-\lambda}^{\lambda} dV(\alpha) + f(0)\, \varepsilon_n(x) \sum_{k=1}^{n} a_k \\
&= 2f(0)\, \varepsilon_n(x)\ .
\end{aligned}$$

Since $\varepsilon_n(x) \leqslant 1$ is always true, recalling Lemma 1 completes the proof.

Corollary 1.1. Any sequence of approximations $S_n(x)$, $n = 1,2, \ldots,$ satisfying Theorem 1 converges uniformly to $f(x)$ on every finite interval.

Proof. Immediate.

Corollary 1.2. If, in addition to the requirements of Theorem 1, $f(x)$ is real valued, then for each integer $n \geqslant 1$, there exists distinct real numbers $\beta_1, \ldots, \beta_n$ and real numbers $d_1, \ldots, d_n$ that satisfy

$$0 < d_k < 1 \ , \ k = 1, \ldots, n \tag{2.11}$$

$$d_1 + d_2 + \ldots + d_n = 1 \tag{2.12}$$

$$0 < \beta_k < \lambda \ , \ k = 1, \ldots, n \tag{2.13}$$

and are such that

$$\left| f(x) - f(0) \sum_{k=1}^{n} d_k \cos \beta_k x \right| \leqslant f(0) \frac{(\lambda x)^{4n}}{2^{4n-2}(4n)!} \tag{2.14}$$

for all $x \varepsilon \mathbf{R}$. Furthermore, the left-hand side of (2.14) is bounded from above by $2f(0)$ for all $x \varepsilon \mathbf{R}$ and every integer $n \geqslant 1$.

Proof. Since $f(x)$ is real valued, by conjugating (1.1) we see that it must be even. From $f(x) = (f(x) + f(-x))/2$ and (2.9), we get

$$f(x) = \int_{-\lambda}^{\lambda} \cos \alpha x \, dV(\alpha) \ . \tag{2.15}$$

Furthermore, the measure $dV(\alpha)$ can be taken to be symmetric about 0. For each $n \geqslant 1$, and for each fixed $x \varepsilon \mathbf{R}$, define

$$\tilde{\varepsilon}_n(x) = \min \max_{-\lambda \leqslant \alpha \leqslant \lambda} |\cos \alpha x - \pi_{2n-1}(\alpha)| \ ,$$

where the minimum is taken over all polynomials $\pi_{2n-1}(\alpha)$ of degree at most $2n-1$ with real coefficients. Hence, we always have $\tilde{\varepsilon}_n(x) \leqslant 1$ by considering the case $\pi_{2n-1}(\alpha) \equiv 0$. From the proof of Lemma 1,

$$\tilde{\varepsilon}_n(x) \leqslant \frac{(\lambda x)^{2n}}{2^{2n-1}(2n)!} \ , \quad x \varepsilon \mathbf{R} \ .$$

Duplicating the proof of Theorem 1 with 2n replacing n gives

$$\left| f(x) - f(0) \sum_{k=1}^{2n} a_k \cos \alpha_k x \right| \leqslant 2f(0)\, \tilde{\varepsilon}_{2n}(x)$$

for the distinct real numbers $\alpha_1, \ldots, \alpha_{2n}$ and real numbers $a_1, \ldots, a_{2n}$ that are the abscissas and Cotes numbers, respectively, of the Gaussian quadrature of order 2n with respect to the measure $dV(\alpha)$. These abscissas and Cotes numbers satisfy

(2.3), (2.4), and (2.5). Since the measure $dV(\alpha)$ is symmetric about zero, it must be that $\alpha_1 = -\alpha_{2n}$, $\alpha_2 = -\alpha_{2n-1}$, etc., and that $a_1 = a_{2n}$, $a_2 = a_{2n-2}$, etc. Inequality (2.14) follows immediately by taking $d_k = 2a_{n+k}$ and $\beta_k = \alpha_{n+k}$ for $k = 1, 2, \ldots, n$. The properties (2.11), (2.12), and (2.13) follow from (2.3), (2.4), and (2.5). This completes the proof.

Example 1. The real valued function

$$f(x) = 2\,\frac{\sin x}{x} = \int_{-1}^{1} e^{i\alpha x}\, dV(\alpha) \tag{2.16}$$

with $V(\alpha) = \alpha$, $-1 \leqslant \alpha \leqslant 1$, is a positive definite function on **R**. In this case, Gaussian quadrature with respect to the measure $dV(\alpha)$ is Gauss-Legendre quadrature. Thus, from the proof of Corollary 1.2, for each $n \geqslant 1$, we have

$$\left| \frac{\sin x}{x} - \sum_{k=1}^{n} d_k \cos \beta_k x \right| \leqslant \frac{x^{4n}}{2^{4n-2}(4n)!}\,, \tag{2.17}$$

where $\beta_1, \ldots, \beta_n$ are the positive abscissas of a 2n point Gauss-Legendre quadrature and $d_1, \ldots, d_n$ are the corresponding Cotes numbers. This example and some computation provides one test of the quality of the error term. Let $R_n(x)$ be the smaller of the two numbers $2f(0) = 2$ and

$$\frac{x^{4n}}{2^{4n-2}(4n)!} \approx \left(\frac{xe}{8n}\right)^{4n} \sqrt{\frac{2}{\pi n}}\,. \tag{2.18}$$

From Table 1, it appears that $R_n(x)$ is an excellent error bound provided $|x| << 8n/e$. (Table 1 was computed on a DEC VAX 11/780 on which the double precision unit round-off error is only 4×10^{-17}.)

Table 1. Comparison of (2.17) for n = 10 ($8n/e \approx 29.43$)

x	$R_{10}(x)$	$f(x) - S_{10}(x)$	$\max\limits_{0\leqslant y\leqslant x} \lvert f(y)-S_{10}(y)\rvert$,
5	$.407 \times 10^{-31}$	underflow	underflow
10	$.447 \times 10^{-19}$	underflow	underflow
15	$.494 \times 10^{-12}$	$.491 \times 10^{-13}$	$.491 \times 10^{-13}$
20	$.491 \times 10^{-7}$	$.164 \times 10^{-8}$	$.164 \times 10^{-8}$
25	$.370 \times 10^{-3}$	$.290 \times 10^{-5}$	$.290 \times 10^{-5}$
30	.543	$.664 \times 10^{-3}$	$.664 \times 10^{-3}$
35	$.200 \times 10^{1}$	$.302 \times 10^{-1}$	$.302 \times 10^{-1}$
40	$.200 \times 10^{1}$	.309	.309
45	$.200 \times 10^{1}$	.501	.569
50	$.200 \times 10^{1}$	-.364	.569

This section is concluded by showing that Prony's method (see, e.g., [14, p. 378] or [15, p. 340]) can be used to compute numerically the approximations of Theorem 1. We need only find the Gaussian quadrature formula of order 2n with respect to the measure $dV(\alpha)$, which is equivalent to solving the equations

$$c_m = \int_{-\lambda}^{\lambda} \alpha^m \, dV(\alpha) = \sum_{k=1}^{n} b_k \, \alpha_k^m \, , \; m = 0,1, \ldots, 2n-1 \tag{2.19}$$

for $b_1, \ldots, b_n$ and $\alpha_1, \ldots, \alpha_n$. Since α_k must be real, write $s_k = \ln \alpha_k$, if $\alpha_k \neq 0$, and $s_k = 0$ if $\alpha_k = 0$. The required equations can now be written as

$$c_m = \sum_{k=1}^{n} b_k \, e^{ms_k} \, , \; m = 0,1, \ldots, 2n-1 \; .$$

This form is precisely the form required for Prony's method. (The use of Prony's method to compute Gaussian quadrature formulas was pointed out to the author by Marvin J. Goldstein.) In principle, we require 2n quadratures, the solutions of two systems of linear equations each of rank n, and the roots of a polynomial of degree n (in this case, all its roots are known to be real, distinct, and have multiplicity one) to compute one approximation for which Theorem 1 holds. Unfortunately, it is known [16] that any procedure that relies upon the moments must become increasingly numerically ill-conditioned as n increases. Fortunately, the use of modified moments (i.e., replacing the α^m in (2.19) with some classical system of orthogonal polynomials) together with an algorithm other than Prony's method often results in a numerically well-conditioned problem for finite intervals. See [17] and [18] for details.

III. Exponential Approximation With Uniform Coefficients

The idea developed in this section for constructing approximations of the form $Q_n(x)$ in which each exponential term enters the approximation with equal weight is basically probabilistic in nature. The integral representation (1.3) of f(x) is approximated by a Riemann sum whose subintervals are equally probable according to the "probability" measure $dV(\alpha)$. In this interpretation, $V(\alpha)$ is a cumulative probability integral that is used to transform n uniformly distributed points in the range of $V(\alpha)$ into n abscissas on the real line distributed according to the measure $dV(\alpha)$. (See [19, p. 314] or [15, p. 389].)

Theorem 2. Let f(x) be a continuous complex valued positive definite function of a real variable such that

$$f(x) = \int_{-\lambda}^{\lambda} e^{i\alpha x}\, dV(\alpha)\ , \quad x\varepsilon\mathbf{R}, \tag{3.1}$$

where V is a continuous and strictly monotone increasing function throughout the finite closed interval $[-\lambda, \lambda]$. Then, for each integer $n \geq 1$, there exists distinct real numbers $\alpha_1, \ldots, \alpha_n$ in the open interval $(-\lambda, \lambda)$ such that

$$|f(x) - Q_n(x)| \leq 2\sqrt{2}\ f(0)\,\lambda|x|/n\ , \quad x\varepsilon\mathbf{R}\ , \tag{3.2}$$

where

$$Q_n(x) = \frac{f(0)}{n}\sum_{k=1}^{n} e^{i\alpha_k x}\ . \tag{3.3}$$

Furthermore, from the remark following (1.6), the left-hand side of (3.2) is bounded from above by 2f(0) for all $x\varepsilon\mathbf{R}$.

Proof. Let the real number x be fixed throughout this proof. Define, for k = 0,1,2, . . . , 2n,

$$\begin{aligned} u_k &= V(-\lambda) + (V(\lambda) - V(-\lambda))\ k/2n \\ v_k &= V^{-1}(u_k)\ . \end{aligned} \tag{3.4}$$

Under the hypotheses on $V(\alpha)$, it is clear that V^{-1} exists and is continuous and strictly monotone on the closed interval $[V(-\lambda), V(\lambda)]$. Hence, the numbers v_k in (3.4) are well defined and are distinct. It will be shown that inequality (3.2) holds for

$$\alpha_k = v_{2k-1}\ ,\ k = 1,2, \ldots, n\ . \tag{3.5}$$

Since

$$V(v_{k+1}) - V(v_k) = (V(\lambda)-V(-\lambda))/2n = f(0)/2n \quad , k = 0,1, \ldots, 2n-1\ , \tag{3.6}$$

it follows from the definition (3.3) of $Q_n(x)$ that

$$Q_n(x) = \sum_{k=1}^{n} \int_{v_{2k-2}}^{v_{2k}} \exp(i\, v_{2k-1} x)\, dV(\alpha) \, . \tag{3.7}$$

By the Mean Value Theorem, there exists ζ_k in the interval between α and v_{2k-1} such that

$$\cos \alpha x - \cos v_{2k-1} x = -x(\alpha - v_{2k-1}) \sin \zeta_k x \, . \tag{3.8}$$

Thus, $\alpha < v_{2k-1}$ implies $\alpha < \zeta_k < v_{2k-1}$ and $v_{2k-1} < \alpha$ implies $v_{2k-1} < \zeta_k < \alpha$. From (3.1) and (3.7),

$$\begin{aligned} \mathrm{Re}(f(x) - Q_n(x)) &= \sum_{k=1}^{n} \int_{v_{2k-2}}^{v_{2k}} (\cos \alpha x - \cos v_{2k-1} x) \;\; dV(\alpha) \\ &= -x \sum_{k=1}^{n} \int_{v_{2k-2}}^{v_{2k}} (\alpha - v_{2k-1}) \sin \zeta_k x \;\; dV(\alpha) \end{aligned} \tag{3.9}$$

and so, taking absolute values,

$$\begin{aligned} |\mathrm{Re}(f(x) - Q_n(x))| &\leqslant |x| \sum_{k=1}^{n} \int_{v_{2k-2}}^{v_{2k}} |\alpha - v_{2k-1}| \, dV(\alpha) \\ &\leqslant |x| \sum_{k=1}^{n} (v_{2k} - v_{2k-2}) (V(v_{2k}) - V(v_{2k-2})) \\ &= 2\lambda \, |x| \, f(0)/n \, . \end{aligned} \tag{3.10}$$

where (3.6) was used in the last step. Similarly,

$$|\mathrm{Im}\,(f(x) - Q_n(x))| \leqslant 2\lambda \, |x| \, f(0)/n \, . \tag{3.11}$$

Clearly, (3.10) and (3.11) together complete the proof.

Corollary 2.1. Any sequence of approximations $Q_n(x)$, $n = 1, 2, \ldots,$ satisfying Theorem 2 converges uniformly to $f(x)$ on every finite interval.

Proof. Immediate.

Corollary 2.2. If, in addition to the requirements of Theorem 2, $f(x)$ is real valued, then for each integer $n \geqslant 1$, there exist distinct real numbers $\beta_1, \ldots, \beta_n$ in the open interval $(0, \lambda)$ such that

$$\left| f(x) - \frac{f(0)}{n} \sum_{k=1}^{n} \cos \beta_k x \right| \leqslant f(0)\, \lambda |x| / n \, . \tag{3.12}$$

Furthermore, the left hand side of (3.12) is never larger than $2f(0)$ for all $x \varepsilon \mathbf{R}$ and integer $n \geqslant 1$.

Proof. Recalling (2.15), follow the proof of Theorem 2 with $2n$ replacing n throughout. Half of the resulting $2n$ α_k's are positive. Set the β_k's equal to the positive α_k's. The details are immediate. This concludes the proof.

The proof of Theorem 2 requires that V be continuous and strictly monotone increasing. It is not clear whether the hypotheses on V can be weakened. On the other hand, the convergence rate of the approximations Q_n can apparently be improved by making further assumptions concerning V. In general, however, better than n^{-2} convergence rates cannot be expected. Consider Example 1, where $V(\alpha)=\alpha$ and $f(x) = x^{-1} \sin x$. From the construction indicated in Corollary 2.2, f(x) is approximated by

$$Q_n(x) = \frac{1}{n} \sum_{k=1}^{n} \cos (2k-1)x/2n = \frac{\sin x}{2n \sin x/2n} . \tag{3.13}$$

It can be shown directly that

$$\frac{|x \sin x|}{24n^2} \leqslant \left| \frac{\sin x}{x} - Q_n(x) \right| \leqslant \frac{x^2}{24n^2} \quad , x \varepsilon \mathbf{R} , \tag{3.14}$$

and

$$\lim_{n\to\infty} n^2[x^{-1} \sin x - Q_n(x)] = \frac{-x \sin x}{24} . \tag{3.15}$$

Hence, in this example, the correct convergence rate is precisely n^{-2} for each fixed x. We point out that the upper bound in (3.14) and the limit (3.15) follow by a trivial application of a suggestive result in Pólya-Szegö [20, Pt. 2, Ch. 1, Pr. 11]. Regrettably, their method seems applicable in this application only to the special measure $V(\alpha) = \alpha$.

A further example seems to indicate that the convergence rate of the approximations Q_n can lie between n^{-1} and n^{-2}.

Example 2. [4, p. 22] Let λ be a finite positive real number. Define V for non-negative arguments $\alpha \geqslant 0$ by

$$V(\alpha) = \begin{cases} \alpha(1-\alpha/2\lambda) & , 0 \leqslant \alpha \leqslant \lambda, \\ \lambda/2 & , \lambda < \alpha, \end{cases}$$

and for negative arguments by the relation $V(-\alpha) = -V(\alpha)$, $\alpha > 0$. Thus, V is an odd function whose derivative $V'(\alpha) = 1 - |\alpha|/\lambda$. Obviously, for all $x \neq 0$,

$$f(x) = \int_{-\lambda}^{\lambda} \cos \alpha x \, dV(\alpha) = \frac{2(1-\cos \lambda x)}{\lambda x^2} ,$$

and $f(0) = \lambda$. Now V^{-1} exists on the interval $[-\lambda, \lambda]$ and, for non-negative arguments, is given by

$$V^{-1}(t) = \lambda(1 - \sqrt{1-2t/\lambda}) , \quad 0 \leqslant t \leqslant \lambda/2 .$$

From the construction indicated in Corollary 2.2,

$$Q_n(x) = \frac{\lambda}{n} \sum_{k=1}^{n} \cos \left[\lambda(1-\sqrt{1-(k-½)/n}\,)x \right], \tag{3.16}$$

and

$$|f(x) - Q_n(x)| \leq \lambda^2 |x|/n \ , \quad x \epsilon \mathbf{R} \ .$$

This estimate is not even close to the truth. In fact, an examination of Table 2 indicates that for sufficiently large n the best error bound may take the form

$$|f(x) - Q_n(x)| \leq \frac{Kx^2}{(2n)^{3/2}} \tag{3.17}$$

for some constant K. In general, we speculate that if V^{-1} satisfies a Lipschitz condition of order r, $0 < r \leq 1$, then the convergence rate is of order $1/n^{1+r}$.

Table 2. Inequality (3.17) for $x = 10, \lambda = 1, K = 10^{-2}$

n	$Kx^2/(2n)^{3/2}$	$f(x) - Q_n(x)$
5	$.316 \times 10^{-1}$	$-.618 \times 10^{-1}$
10	$.112 \times 10^{-1}$	$-.117 \times 10^{-1}$
20	$.395 \times 10^{-2}$	$-.153 \times 10^{-2}$
40	$.140 \times 10^{-2}$	$.649 \times 10^{-4}$
80	$.494 \times 10^{-3}$	$.163 \times 10^{-4}$
160	$.175 \times 10^{-3}$	$.907 \times 10^{-4}$
320	$.618 \times 10^{-4}$	$.400 \times 10^{-4}$
640	$.218 \times 10^{-4}$	$.160 \times 10^{-4}$
1280	$.772 \times 10^{-5}$	$.614 \times 10^{-5}$
2560	$.273 \times 10^{-5}$	$.229 \times 10^{-5}$
5120	$.965 \times 10^{-6}$	$.837 \times 10^{-6}$
10240	$.341 \times 10^{-6}$	$.303 \times 10^{-6}$
20480	$.121 \times 10^{-6}$	$.109 \times 10^{-6}$
40960	$.426 \times 10^{-7}$	$.389 \times 10^{-7}$

IV. Concluding Remarks

The proofs in this report depend heavily on the finite support of the measure $dV(\alpha)$ even though the construction of the approximations $S_n(x)$ and $Q_n(x)$ can be carried out without modification on infinite intervals as well, provided $V(\alpha)$ is bounded. Since these proofs cannot be adapted for infinite intervals, the effectiveness of the resulting approximations theoretically remains an open question. Intuitively, however, it would seem that only our proofs are limited and that the underlying approximation process is generally valid.

In computational practice the function V is usually unknown. In many cases, however, the given function f does possess a nicely behaved Fourier transform from which V can be readily constructed. The Fourier transform of f can, of course, be computed accurately and efficiently in many situations using fast Fourier transform (FFT) methods.

If in the above, V was not monotonic, but of bounded variation on **R**, exponential approximations can be constructed as follows. In this case, there exist monotone increasing functions V^+ and V^- such that $V = V^+ - V^-$. For each $\lambda > 0$, define the "bandlimiting" operator B_λ by

$$B_\lambda f(x) = \int_{-\lambda}^{\lambda} e^{i\alpha x} dV(\alpha) , \quad x \epsilon \mathbf{R} .$$

Let I be any finite interval. Given $\varepsilon > 0$, choose $\lambda > 0$ so that $||f - B_\lambda f|| < \varepsilon$, where the norm is the uniform norm over the interval I. Now, let the two functions

$$B_\lambda^{\pm} f(x) \equiv \int_{-\lambda}^{\lambda} e^{i\alpha x} dV^{\pm}(\alpha) , \quad x \epsilon \mathbf{R} ,$$

be approximated using either of the methods of this paper by the two finite exponential sums, say, $E^{\pm}(x)$, so that

$$||B_\lambda^{\pm} f - E^{\pm}|| < \varepsilon .$$

Let $E(x) = E^+(x) - E^-(x)$. Since $B_\lambda f = B_\lambda^+ f - B_\lambda^- f$, we have

$$\begin{aligned} ||f - E|| &= ||(f - B_\lambda f) + (B_\lambda^+ f - E^+) - (B_\lambda^- f - E^-)|| \\ &\leq ||f - B_\lambda f|| + ||B_\lambda^+ f - E^+|| + ||B_\lambda^- f - E^-|| < 3\varepsilon . \end{aligned}$$

Thus, exponential sums of degree not greater than $\deg(E^+) + \deg(E^-)$ may be constructed to approximate f(x) with specified accuracy on the interval I.

It is well known [4, p. 60] that if f(x) is measurable, then Bochner's Theorem still holds for almost all x, but not necessarily all x as in (1.2). Results similar to the results of this report can also be proven for measurable f with careful attention to certain details; however, this generalization is not pursued here. Similarly, Bochner's Theorem has been generalized to locally compact abelian groups [4, p. 72], and perhaps the basic approaches to approximation used here can be extended to this much more general setting.

We conclude by commenting that Bernstein's Theorem [21, p. 160] states that a necessary and sufficient condition for f(x) to be completely monotonic on the interval $(0,\infty)$ is that

$$f(x) = \int_0^\infty e^{-\alpha x}\, dV(\alpha)\,, \quad 0 \leqslant x < \infty\,,$$

where $V(\alpha)$ is bounded and nondecreasing. It is evident that the methods employed in this report can be used in a manner entirely analogous to the proof of Theorem 1 to develop exponential approximations. That is to say, whenever f(x) can be expressed as

$$f(x) = \int_0^\lambda e^{-\alpha x}\, dV(\alpha)$$

for some finite $\lambda > 0$, there exist approximations of the form

$$T_n(x) = f(0) \sum_{k=1}^{n} a_k\, e^{-\alpha_k x}$$

where

$$a_k > 0\,,\; k = 1, \ldots, n \tag{4.1}$$

$$a_1 + \ldots + a_n = 1 \tag{4.2}$$

$$\lambda > \alpha_k > 0\,,\; k = 1, \ldots, n \tag{4.3}$$

and

$$|f(x) - T_n(x)| \leqslant 2f(0)\, \frac{(\lambda x)^{2n}}{2^{2n}}\, e^{-\lambda x/2}, \quad 0 \leqslant x < \infty\,. \tag{4.4}$$

An alternate approach for approximation of completely monotonic functions can be found in [22].

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